MATH 4901 Final Report

2 Dimensional and General Proofs to Isoperimetric Inequality

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1. Introduction

There was a well-posed question in my elementary school:

> What shape maximizes the area of yard enclosed by a total of x feet of fencing?

This problem naturally arises from real life. It is also widely studied in mathematics under the name ”Isoperimetric Inequality”. The problem is stated in either ways below:

(1) Maximize the area bounded by a positively oriented simple closed curve.
(2) Minimize the length of a positively oriented simple closed curve to bound a fixed measure of area.

People knew the answer(circle) since the Greek times, but a formal proof took a lot more time to be found.

Here we present a formal version of the theorem, and two proofs: one specific to plane curves using classic differential geometry tools and one general using concepts in measure theory.

2. Isoperimetric Inequality For Plane Curves

First we present the plane curve version by do Carmo. This is the case that is closest to the fencing problem stated above. Formally the theorem is stated as below:

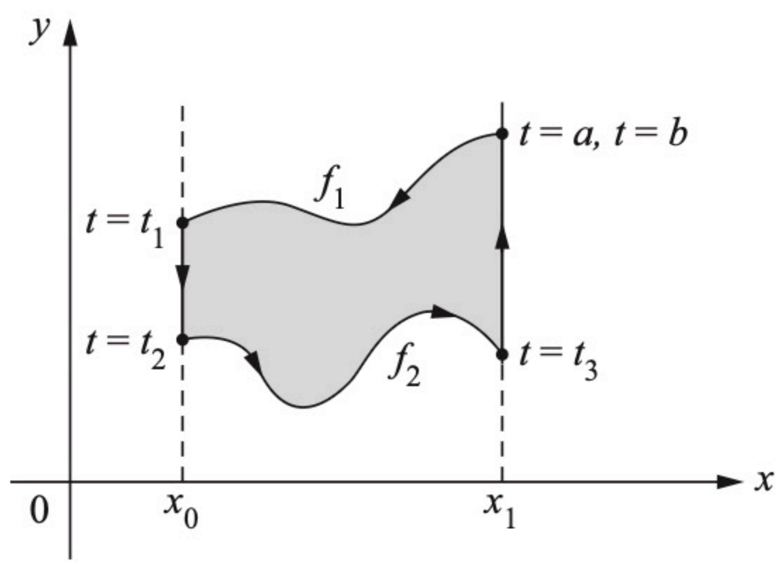
Theorem 2.1. *(Isoperimetric Inequality for plane curves) Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then*

$$l^2 - 4\pi A \geq 0.$$

Proof. Consider a positively oriented simple closed curve $\alpha(t) = (x(t), y(t))$, where $t \in [a, b]$ is the parametrization variable.

We claim that it is possible to divide the region enclosed by the curve into a finite number of disjoint regions that is enclosed by two straight line segments parallel to y axis and two arbitrary simple curves f_1 and f_2.

We use the image in do Carmo's book.



To compute the shaded area, we can use the integral of $f_1 - f_2$.

$$A = \int_{x_0}^{x_1} (f_1(x) - f_2(x))dx = \int_{x_0}^{x_1} f_1(x)dx - \int_{x_0}^{x_1} f_2(x)dx.$$

If we parametrize the curve using t, keeping in mind $x'(t) = 0$ along vertical segments, we have:

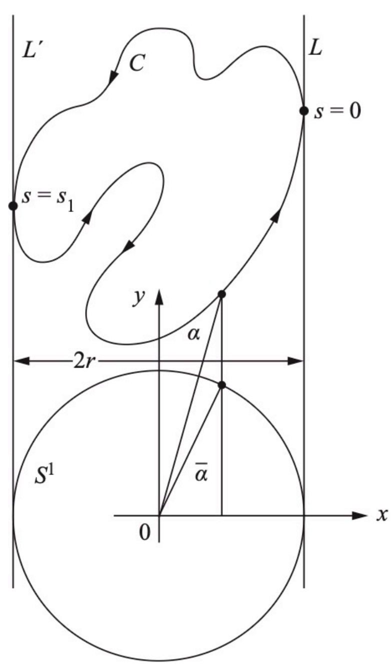
$$A = -\int_a^{t_1} y(t)dx(t) - \int_{t_2}^{t_3} y(t)dx(t) = -\int_a^b y(t)x'(t)dt.$$

Using integration by parts formula, this can also be written as:

$$\begin{aligned} A = -\int_a^b y(t)x'(t)dt &= \int_a^b (xy)'dt - \int_a^b x'(t)y(t)dt \\ &= [xy(b) - xy(a)] - \int_a^b x'ydt \\ &= \int_a^b x'ydt. \end{aligned}$$

Then we proceed with the proof. Let L and L' be two parallel straight lines that are both tanegent to C. Denote the shortest distance between the two straight lines $2r$. Draw a circle S_1 of radius r below C, let O be its origin. Adjust the position of S_1 so that S_1 and C does not intersect.

Then set the axis frames such that O is the origin, x axis is perpenticular to both L and L', and y axis is parallel to the two straight lines. Let $\alpha(s) = (x(s), y(s)), s \in [0, l]$ be the parametrization for C, and let $\bar{\alpha} = (\bar{x}(s), \bar{y}(s)), s \in [0, l]$ be the parametrization for S_1. Denote the tangent points of L and L' with C as $s = 0$ and $s = s_1$.



Using the previously derived formula for area, we denote $\bar{A}$ to be the area enclosed by S_1.

$$A = \int_0^l xy' ds, \quad \bar{A} = -\int_0^l \bar{y}x' ds = \pi r^2.$$

Adding both parts together, we have:

$$\begin{aligned} A + \pi r^2 &= \int_0^l (xy' - \bar{y}x') ds \leq \int_0^l \sqrt{(xy' - \bar{y}x)^2} ds \\ &\leq \int_0^l \sqrt{(x^2 + \bar{y}^2)((x')^2 + (y')^2)} ds = \int_0^l \sqrt{\bar{x}^2 + \bar{y}^2} ds \\ &= lr. \end{aligned}$$

By the AM-GM Inequality, the arithmetic mean is always greater than or equal to the geometric equality. Thus,

$$\sqrt{A}\sqrt{\pi r^2} \leq \frac{1}{2}(A + \pi r^2) \leq \frac{1}{2} lr$$

$$4\pi A r^2 \leq l^2 r^2.$$

This is indeed an elegant proof, and this is why I included this proof even when this is only a subset of a general proof.

In fact, the Isoperimetric Inequality we proved here works as long as:

(1) The arclength is well defined.
(2) The area enclosed is measurable.
(3) The set of corners has Lebesgue measure zero.

This seems pretty general and motivates generalizing of the proof in the next section. □

3. General Isoperimetric Inequality

Definition 3.1. (Minkowski Sum) For two compact sets A and B in the Euclidean space $\mathbb{R}^n$, define the Minkowski Sum as following:

$$A + B = \{a + b | a \in A, b \in B\} = \bigcup_{a \in A} (a + B)$$

Definition 3.2. (Volume) The volume $\mathrm{Vol}(A)$ of a compact set $A \subseteq \mathbb{R}^n$ is defined as its n-dimensional Lebesgue measure, denoted as $\lambda_n(A)$. The measure has the following properties:

(1) For the unit cube $Q = [0,1]^n$, $\lambda_n(Q) = 1$.
(2) $\lambda_n(A + p) = \lambda_n(A) \forall p \in \mathbb{R}^n$.
(3) For disjoint measurable sets $\{A_i\}$, $\lambda_n(\bigcup_i A_i) = \sum_i \lambda_n(A_i)$.

Note that the third property relies on regular addition operation while the below theorem is talking about the Minkowski Sum defined above.

Theorem 3.3. *(Brunn-Minkowski Inequality) Let A and B be two non empty compact sets in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}(A + B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n},$$

The general proof is a bit tricky here. So we try to prove a simple case, and then try to project the general case onto the simple case. The case we choose is called the brick set.

Definition 3.4. (Brick Set) A set $A \in \mathbb{R}^n$ is a brick set if it is the union of finitely many closed axis parallel boxes with disjoint interiors.

Lemma 3.5. *(Brunn-Minkowski Inequality for Brick Sets) Let A and B be two non empty brick sets in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}(A + B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}.$$

Proof. We perform a proof by induction on the number of bricks k here.
Base case:

Choose $k = 2$. In other words, A and B are single bricks. Let $A = \prod_{i=1}^n [0, a_i]$ and $B = \prod_{i=1}^n [0, b_i]$. The Minkowski Sum $A + B$ is the brick $\prod_{i=1}^n [0, a_i + b_i]$.

Then the volumes are:

$$\mathrm{Vol}(A) = \prod_{i=1}^n a_i, \quad \mathrm{Vol}(B) = \prod_{i=1}^n b_i, \quad \mathrm{Vol}(A + B) = \prod_{i=1}^n (a_i + b_i).$$

Substituting into the inequality we are trying to prove:

$$\prod_{i=1}^{n}(a_i+b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n}.$$

Divide both sides by $\prod_{i=1}^{n}(a_i+b_i)^{1/n}$:

$$1 \geq \prod_{i=1}^{n}\left(\frac{a_i}{a_i+b_i}\right)^{1/n} + \prod_{i=1}^{n}\left(\frac{b_i}{a_i+b_i}\right)^{1/n}.$$

Divide both sides by $(\prod_{i=1}^{n}(a_i+b_i))^{1/n}$:

$$1 \geq \prod_{i=1}^{n}\left(\frac{a_i}{a_i+b_i}\right)^{1/n} + \prod_{i=1}^{n}\left(\frac{b_i}{a_i+b_i}\right)^{1/n}.$$

Let $\theta_i = \frac{a_i}{a_i+b_i}$ and $1-\theta_i = \frac{b_i}{a_i+b_i}$. Define the geometric means:

$$G_A = \prod_{i=1}^{n}\theta_i^{1/n}, \quad G_B = \prod_{i=1}^{n}(1-\theta_i)^{1/n}.$$

By the AM-GM inequality for each $\theta_i \in [0,1]$:

$$G_A \leq \frac{1}{n}\sum_{i=1}^{n}\theta_i \quad \text{and} \quad G_B \leq \frac{1}{n}\sum_{i=1}^{n}(1-\theta_i).$$

Summing these inequalities:

$$G_A + G_B \leq \frac{1}{n}\sum_{i=1}^{n}\theta_i + \frac{1}{n}\sum_{i=1}^{n}(1-\theta_i) = \frac{1}{n}\sum_{i=1}^{n}\left(\theta_i + (1-\theta_i)\right) = \frac{1}{n}\sum_{i=1}^{n}1 = 1.$$

Thus:

$$G_A + G_B \leq 1 \quad \Longrightarrow \quad \mathrm{Vol}(A+B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}.$$

Inductive Hypothesis:

Assume the inequality holds for all brick sets with a total of $k-1$ bricks.

Inductive Step:

Let A, B be brick sets with k total bricks. Without loss of generality, assume A contain ≥ 2 bricks.

Since A is a finite union of disjoint bricks, there exists a coordinate hyperplane $H = \{x_j = c\}$ that splits $\mathbb{R}^n$ into halfspaces H^+ and H^- such that:

$$\mathrm{Vol}(A^+) = \rho\,\mathrm{Vol}(A) \quad \text{and} \quad \mathrm{Vol}(A^-) = (1-\rho)\,\mathrm{Vol}(A),$$

where $0 < \rho < 1$, and $A^{\pm} = A \cap H^{\pm}$ are brick sets with fewer bricks than A. By the intermediate value theorem, we may translate B such that the same hyperplane H splits B proportionally:

$$\operatorname{Vol}(B^+) = \rho \operatorname{Vol}(B) \quad \text{and} \quad \operatorname{Vol}(B^-) = (1-\rho)\operatorname{Vol}(B).$$

Both $A^{\pm} + B^{\pm}$ have $\leq k-1$ bricks. By the induction hypothesis:

$$\operatorname{Vol}(A^+ + B^+)^{1/n} \geq \operatorname{Vol}(A^+)^{1/n} + \operatorname{Vol}(B^+)^{1/n},$$
$$\operatorname{Vol}(A^- + B^-)^{1/n} \geq \operatorname{Vol}(A^-)^{1/n} + \operatorname{Vol}(B^-)^{1/n}.$$

The sets $A^+ + B^+$ and $A^- + B^-$ are disjoint and lie in H^+ and H^- respectively. Thus, their volumes are additive in $A + B$:

$$\operatorname{Vol}(A+B) \geq \operatorname{Vol}(A^+ + B^+) + \operatorname{Vol}(A^- + B^-).$$

Substituting the inductive bounds:

$$\begin{aligned}\operatorname{Vol}(A+B) \geq & \left[(\rho \operatorname{Vol}(A))^{1/n} + (\rho \operatorname{Vol}(B))^{1/n}\right]^n \\ & + \left[((1-\rho)\operatorname{Vol}(A))^{1/n} + ((1-\rho)\operatorname{Vol}(B))^{1/n}\right]^n.\end{aligned}$$

Factoring $\rho^{1/n}$ and $(1-\rho)^{1/n}$:

$$\operatorname{Vol}(A+B) \geq \rho\left[\operatorname{Vol}(A)^{1/n} + \operatorname{Vol}(B)^{1/n}\right]^n + (1-\rho)\left[\operatorname{Vol}(A)^{1/n} + \operatorname{Vol}(B)^{1/n}\right]^n.$$

$$\operatorname{Vol}(A+B) \geq \left[\operatorname{Vol}(A)^{1/n} + \operatorname{Vol}(B)^{1/n}\right]^n.$$

$$\operatorname{Vol}(A+B)^{1/n} \geq \operatorname{Vol}(A)^{1/n} + \operatorname{Vol}(B)^{1/n}. \qquad \square$$

Proof. (Proof to Theorem 3.3 Brunn-Minkowski Inequality)

Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$ be a sequence of finite brick sets with $\bigcup_i A_i = A$, and let $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_i \subseteq \cdots$ be a sequence of finite brick sets with $\bigcup_i B_i = B$. We have:

$$\lim_{i\to\infty} \mathrm{Vol}(A_i) = \mathrm{Vol}(A) \quad \text{and} \quad \lim_{i\to\infty} \mathrm{Vol}(B_i) = \mathrm{Vol}(B).$$

Then for any $z \in A + B$, there exist $u \in A$ and $v \in B$ such that $z = u + v$. By construction, there exists i such that $u \in A_j$ and $v \in B_j$ for all $j \geq i$, hence $z \in A_j + B_j$. This implies:

$$A + B \subseteq \bigcup_j (A_j + B_j) \subseteq \bigcup_j (A + B) = A + B,$$

so $\bigcup_j (A_j + B_j) = A + B$. The monotonicity of volume and continuity under nested unions then yield:

$$\lim_{i\to\infty} \mathrm{Vol}(A_i + B_i) = \mathrm{Vol}(A + B).$$

By Lemma 3.5, the Brunn-Minkowski inequality holds for all finite brick sets:

$$\mathrm{Vol}(A_i + B_i)^{1/n} \geq \mathrm{Vol}(A_i)^{1/n} + \mathrm{Vol}(B_i)^{1/n}.$$

Taking the limit as $i \to \infty$, the continuity of the $(\cdot)^{1/n}$ function preserves the inequality:

$$\begin{aligned} \mathrm{Vol}(A + B)^{1/n} &= \lim_{i\to\infty} \mathrm{Vol}(A_i + B_i)^{1/n} \\ &\geq \lim_{i\to\infty} \left(\mathrm{Vol}(A_i)^{1/n} + \mathrm{Vol}(B_i)^{1/n}\right) \\ &= \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}. \end{aligned}$$

□

Theorem 3.6. *(Isoperimetric inequality) Let $K \subset \mathbb{R}^n$ be a convex body and let $b \subset \mathbb{R}^n$ be the unit ball. Then:*

$$\left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(b)}\right)^{1/n} \leq \left(\frac{S(K)}{S(b)}\right)^{1/(n-1)},$$

where $S(X)$ denotes the surface area of X.

Proof. For $\epsilon > 0$, the Minkowski sum $K + \epsilon b$ expands K uniformly by ϵ. The surface area $S(K)$ can be expressed as:

$$S(K) = \lim_{\epsilon\to 0^+} \frac{\mathrm{Vol}(K + \epsilon b) - \mathrm{Vol}(K)}{\epsilon}.$$

By the Brunn-Minkowski inequality:

$$\mathrm{Vol}(K+\epsilon b)^{1/n} \geq \mathrm{Vol}(K)^{1/n} + \mathrm{Vol}(\epsilon b)^{1/n}.$$

Raising both sides to the nth power:

$$\mathrm{Vol}(K+\epsilon b) \geq \left(\mathrm{Vol}(K)^{1/n} + \epsilon\,\mathrm{Vol}(b)^{1/n}\right)^n.$$

By the binomial theorem for $\epsilon \to 0$:

$$\mathrm{Vol}(K+\epsilon b) \geq \mathrm{Vol}(K) + n\epsilon\,\mathrm{Vol}(K)^{(n-1)/n}\,\mathrm{Vol}(b)^{1/n} + O(\epsilon^2).$$

Substitute back into the surface area definition:

$$\begin{aligned} S(K) &\geq \lim_{\epsilon\to 0^+} \frac{\mathrm{Vol}(K) + n\epsilon\,\mathrm{Vol}(K)^{(n-1)/n}\,\mathrm{Vol}(b)^{1/n} - \mathrm{Vol}(K)}{\epsilon} \\ &= n\,\mathrm{Vol}(K)^{(n-1)/n}\,\mathrm{Vol}(b)^{1/n}. \end{aligned}$$

Since:

$$\mathrm{Vol}(b) = \int_0^1 S(rb)\,dr = \int_0^1 n\,\mathrm{Vol}(b) r^{n-1} dr = n\,\mathrm{Vol}(b)\cdot\frac{1}{n} = \mathrm{Vol}(b),$$

$S(b) = n\,\mathrm{Vol}(b)$. Substitute $S(b) = n\,\mathrm{Vol}(b)$ back into the surface area bound:

$$\frac{S(K)}{S(b)} \geq \frac{\mathrm{Vol}(K)^{(n-1)/n}\,\mathrm{Vol}(b)^{1/n}}{\mathrm{Vol}(b)} = \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(b)}\right)^{(n-1)/n}.$$

Taking the $(n-1)$-th root:

$$\left(\frac{S(K)}{S(b)}\right)^{1/(n-1)} \geq \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(b)}\right)^{1/n}.$$

□

4. Conclusion

Although both proofs converge to the same statement, the complexity and generality differs. We can see the power of measure theory techniques and its help to geometric analysis from the difference. While the planar case relies on intuitive geometric arguments and careful construction of integrands, the Brunn-Minkowski framework extends the result to arbitrary dimensions with rigorous generality.

References

[1] Manfredo P. do Carmo. *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*. Dover Publications, Mineola, NY, 2 edition, 2016.

[2] Sariel Har-Peled. Chapter 22: Brunn-minkowski inequality. Lecture notes from CS 498SH3, University of Illinois at Urbana-Champaign.