The Fary-Milnor Theorem

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1. Preliminaries

In this independent study paper, we aim to derive a self contained proof to the *Fary-Milnor Theorem* 4.8.

Definition 1.1. For a closed polygon P with vertices $a_0, a_1, \dots, a_m = a_0$, let α_i be the angle between the vectors $a_{i+1} - a_i$ and $a_i - a_{i-1}$ satisfying $0 \leq \alpha_i \leq \pi$. The total curvature of P is defined as:

$$\kappa(P) = \sum_{i=1}^{m} \alpha_i.$$

Lemma 1.2. *If a closed polygon P has four consecutive non-coplanar vertices $a_{j-2}, a_{j-1}, a_j, a_{j+1}$, and the vertex a_j is replaced by a point a'_j on the line segment $a_j a_{j+1}$, then the resulting polygon P' satisfies*

$$\kappa(P') < \kappa(P).$$

Remark 1.3. In other words, adding a vertex will generally strictly increase the total curvature.

Definition 1.4. A simple closed curve C in $\mathbb{R}^3$ is an unknot if there exists an isotopy of $\mathbb{R}^3$ that transforms C into a circle. Otherwise, C is a knot.

Definition 1.5. A **closed curve** in $\mathbb{R}^n$ is described by a continuous vector function $\mathbf{r}(t)$ of period L:

$$\mathbf{r}(t) = (r_1(t), \dots, r_n(t)).$$

The curve is simple if $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ only when $(t_1 - t_2)/L$ is an integer.

Definition 1.6. A polygon P is inscribed in the curve if there exist parameters $t_1 < \cdots < t_m$ such that the vertices of P are $\mathbf{r}(t_1), \dots, \mathbf{r}(t_m)$.

2. Closed Curves in Euclidean Space

Lemma 2.1. *For any closed polygon P, the total curvature $k(P)$ satisfies:*

$$k(P) = \sup\{k(P') : P' \text{ is a polygon inscribing } P\}.$$

Proof. The inequality $k(P) \geq k(P')$ follows from the fact that inscribing additional vertices can only increase the total curvature. To show equality, consider a sequence of inscribed polygons P_n with vertices becoming dense in P. By continuity of the curve, the angles between successive segments of P_n approach those of P, hence $k(P_n) \to k(P)$. □

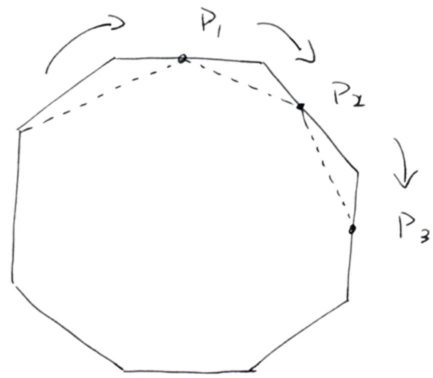

Figure 1

Theorem 2.2. *If C is a closed curve of class C^2 parameterized by arc length s, then:*

$$k(C) = \int_C |\mathbf{r}''(s)| ds.$$

Proof. Let $\{s_1^m, \ldots, s_m^m\}$ be the vertices of a polygon P_m inscribed in C such that:

$$\lim_{m\to\infty} \max_i |s_{i+1}^m - s_i^m| = 0.$$

Define the midpoint $\bar{s}_i^m = (s_i^m + s_{i+1}^m)/2$ and let θ_i^m be the angle between $\mathbf{r}'(s_i^m)$ and $\mathbf{r}'(s_{i+1}^m)$.

The vectors $\mathbf{r}'(s)$ describe a curve L on the unit sphere with length:

$$\int_C |\mathbf{r}''(s)| ds.$$

The vectors $\mathbf{r}'(s_i^m)$ form vertices of a spherical polygon inscribed in L, hence:

$$\lim_{m\to\infty}\sum_{i=1}^{m}\theta_i^m = \int_C |\mathbf{r}''(s)|ds.$$

Since $\mathbf{r}''$ is uniformly continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that:

$$|\mathbf{r}''(u) - \mathbf{r}''(v)| < \epsilon \quad \text{whenever } |u - v| < \delta.$$

We analyze the difference:

$$\mathbf{r}(s_{i+1}^m) - \mathbf{r}(s_i^m) = (s_{i+1}^m - s_i^m)\mathbf{r}'(\bar{s}_i^m) + R_i^m,$$

where the remainder term R_i^m satisfies the estimate:

$$|R_i^m| \leq \frac{(s_{i+1}^m - s_i^m)^2\epsilon}{2}.$$

Let ϕ_i^m be the angle between $\mathbf{r}(s_{i+1}^m) - \mathbf{r}(s_i^m)$ and $\mathbf{r}'(s_i^m)$. Then:

$$\sin\phi_i^m \leq \frac{|R_i^m|}{s_{i+1}^m - s_i^m} \leq \frac{(s_{i+1}^m - s_i^m)\epsilon}{2}.$$

For sufficiently small ϵ, we have $\phi_i^m \leq (s_{i+1}^m - s_i^m)\epsilon$.

Let α_i^m be the angle between successive segments of P_m. By the triangle inequality for angles:

$$|\alpha_i^m - \theta_i^m| \leq \phi_i^m + \phi_{i+1}^m.$$

Summing over i:

$$\left|\sum_{i=1}^{m}\alpha_i^m - \sum_{i=1}^{m}\theta_i^m\right| \leq 2l\epsilon,$$

where l is the length of C. Since ϵ was arbitrary:

$$\lim_{m\to\infty} k(P_m) = \lim_{m\to\infty}\sum_{i=1}^{m}\alpha_i^m = \lim_{m\to\infty}\sum_{i=1}^{m}\theta_i^m = \int_C |\mathbf{r}''(s)|ds.$$

□

3. The Crookedness of a Closed Curve

Definition 3.1. For a closed curve C parameterized by $\mathbf{r}(t)$ and a unit vector $b \in S^{n-1}$, define $\mu(C, b)$ to be the number of local maxima of the function $b \cdot \mathbf{r}(t)$ in a fundamental period. The crookedness of C is defined as:

$$\mu(C) = \min_{b \in S^{n-1}} \mu(C, b).$$

Remark 3.2. In other words, $\mu(C, b)$ counts the number of local maximums in principle direction of b. The crookedness $\mu(C)$ is the minimum this count over all possible directions.

Theorem 3.3. *For any closed curve C in $\mathbb{R}^n$ with $n \geq 2$, the Lebesgue integral exists and*

$$\int_{S^{n-1}} \mu(C, b) dS(b) = \frac{M_{n-1}}{2\pi} \kappa(C).$$

Where $M_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface measure of S^{n-1}.

We need the following lemmas to prove Theorem 3.3.

Let P be a closed polygon with vertices $a_1, \ldots, a_m$ and edges $a_i a_{i+1}$. Normalize and define the unit vectors:

$$b_i = \frac{a_{i+1} - a_i}{|a_{i+1} - a_i|}.$$

These vectors lie on the unit sphere S^{n-1} and they are the directions of the polygon's edges.

Lemma 3.4. *For a closed polygon P, join each b_{i-1} to b_i by a great circle arc of length α_i (the exterior angle at a_i). This forms a closed spherical polygon Q on S^{n-1} of total length $\kappa(P)$.*

Proof. The length of each great circle arc is just the exterior angle α_i between consecutive edges. So total length of Q is $\sum_{i=1}^{m} \alpha_i = \kappa(P)$. □

Lemma 3.5. *For a closed polygon P and a unit vector b, the number of intersections of the spherical image Q with the great sphere $S_b^{n-2} = \{\mathbf{x} \in S^{n-1} : \mathbf{x} \cdot b = 0\}$ equals $2\mu(P, b)$, if no edge of P is perpendicular to b.*

Proof. An edge $b_{i-1} b_i$ of Q crosses S_b^{n-2} if and only if $b \cdot (a_i - a_{i-1})$ and $b \cdot (a_{i+1} - a_i)$ have opposite signs. This occurs exactly when $b \cdot \mathbf{r}(t)$ has a local maximum or minimum at the vertex a_i. Since maxima and minima alternate for a polygon, the total number of crossings is $2\mu(P, b)$. □

Proof of Theorem 3.3 for Polygons. Let P be a closed polygon.

Consider the set of directions b for which no edge of P is perpendicular to b. This will exclude a finite union of great spheres, which has measure zero in S^{n-1}.

For such b, by Lemma 3.5, the spherical image Q intersects S_b^{n-2} exactly $2\mu(P,b)$ times.

Now, we recall the Cauchy-Crofton formula for S^2, the integral over S^{n-1} of the number of intersections of Q with S_b^{n-2} equals $\frac{2}{\sigma_{n-2}} \cdot \text{length}(Q)$, where σ_{n-2} is the surface measure of S^{n-2}.

Moreover, for a fixed great sphere S_b^{n-2}, the measure of directions b for which S_b^{n-2} intersects a given great circle arc of length α_i is proportional to α_i.

Actually, the contribution from the i-th edge is:

$$\int_{S^{n-1}} \chi_{\{b: S_b^{n-2} \cap b_{i-1}b_i \neq \emptyset\}} dS(b) = \frac{M_{n-1}}{\pi} \alpha_i.$$

Summing over all edges and recall that each intersection corresponds to either a maximum or minimum (and the numbers are equal), we have:

$$\int_{S^{n-1}} 2\mu(P,b) dS(b) = \frac{M_{n-1}}{\pi} \sum_{i=1}^{m} \alpha_i = \frac{M_{n-1}}{\pi} \kappa(P).$$

$$\int_{S^{n-1}} \mu(P,b) dS(b) = \frac{M_{n-1}}{2\pi} \kappa(P).$$

□

We want to use approximation to generalize this argument.

Lemma 3.6. *If $\{P_m\}$ is a sequence of inscribed polygons in C with*

$\lim_{m\to\infty} \kappa(P_m) = \kappa(C)$ *and* $\lim_{m\to\infty} \max_i |t_i^{m+1} - t_i^m| = 0$*, then for a.e.* $b \in S^{n-1}$*:*

$$\mu(C,b) = \lim_{m\to\infty} \mu(P_m, b).$$

Proof. Fix b, then $\mu(P_m,b)$ is non-decreasing in m since adding vertices can only increase the number of local maxima.

If $\mu(C,b) < \infty$, then we can find neighborhood around each of the $\mu(C,b)$ maxima such that any sufficiently fine inscribed polygon will have at least one vertex in each neighborhood, hence $\mu(P_m,b) \geq \mu(C,b)$ for large m.

If $\mu(C,b) = \infty$, then there are infinitely many local maximas. For all $N > 0$, we can find N disjoint neighborhoods containing local maxima, and for sufficiently large m, the polygon P_m will have at least one vertex in each neighborhood, so $\mu(P_m,b) \geq N$.

The problematic set where the limit might fail consists of directions b for which $b \cdot \mathbf{r}(t)$ or other measure zero cases, which has measure zero. □

Proof of Theorem 3.3 for General Curves. Let $\{P_m\}$ be a sequence of inscribed polygons with:

$$\lim_{m\to\infty} \kappa(P_m) = \kappa(C) \text{ and } \lim_{m\to\infty} \max_i |t_i^{m+1} - t_i^m| = 0.$$

By the monotone convergence theorem and Lemma 3.6, we have:

$$\begin{aligned}\int_{S^{n-1}} \mu(C, bdS(b &= \lim_{m\to\infty} \int_{S^{n-1}} \mu(P_m, bdS(b \\ &= \lim_{m\to\infty} \frac{M_{n-1}}{2\pi}\kappa(P_m) = \frac{M_{n-1}}{2\pi}\kappa(C).\end{aligned}$$

□

As an immediate corollary of Theorem 3.3, we have the following fundamental relationship between total curvature and crookedness:

Theorem 3.7. *For any closed curve C in $\mathbb{R}^n$ with $n \geq 2$:*

$$\kappa(C) \geq 2\pi\mu(C).$$

Proof. From Theorem 3.3, we have:

$$\frac{M_{n-1}}{2\pi}\kappa(C) = \int_{S^{n-1}} \mu(C, bdS(b \geq \int_{S^{n-1}} \mu(C)dS(b = M_{n-1}\mu(C).$$

Dividing both sides by $M_{n-1}/2\pi$ will give the desired result. □

Corollary 3.8. *For any closed curve C in $\mathbb{R}^3$, $\kappa(C) \geq 2\pi$.*

Proof. $\mu(C) \geq 1$ for any closed curve, 3.7 gives the immediate result. □

4. Curvature and Crookedness of Isotopy Types

Definition 4.1. Two closed curves C and C' in $\mathbb{R}^n$ are equivalent by isotopy if there exists a continuous family of homeomorphisms $F_t : \mathbb{R}^n \to \mathbb{R}^n$ for $0 \le t \le 1$ such that F_0 is the identity and $F_1(C) = C'$.

Definition 4.2. A curve type G is an equivalence class of closed curves under isotopy. A curve type is simple if its representatives are simple closed curves. A simple curve type is unknotted if it contains all circles; otherwise it is knotted.

Definition 4.3. A curve type G is tame if it contains a polygon; otherwise it is wild.

Definition 4.4. For a curve type G, define:

$$\kappa(G) = \inf\{\kappa(C) : C \in G\}, \quad \mu(G) = \min\{\mu(C) : C \in G\}.$$

Theorem 4.5. *For any simple closed curve C with $\mu(C) < \infty$, there exists a polygon P inscribed in C that is equivalent to C by isotopy.*

Proof. Let b be a unit vector such that $\mu(C, b) = \mu(C) < \infty$. Let $t_1 < t_2 < \cdots < t_{2\mu(C)}$ be the parameter values where $b \cdot \mathbf{r}(t)$ attains local maxima and minima.

Around each point $\mathbf{r}(t_i)$, construct a cylinder Z_i with axis parallel to b such that:

- Z_i intersects C in exactly two points $\mathbf{r}(t_i^-)$ and $\mathbf{r}(t_i^+)$
- Both intersection points lie on the same base of the cylinder
- $\mathbf{r}(t_i^-)$ is the center of this base

We now perform an isotopy in three stages:

1: Within each cylinder Z_i, we isotope the arc $\mathbf{r}(t)$ for $t_i^- \le t \le t_i^+$ to the straight line segment $\mathbf{r}(t_i^-)\mathbf{r}(t_i)\mathbf{r}(t_i^+)$. We do the following:

(1) Project the arc onto the cylinder's axis in hyperplanes perpendicular to b,
(2) Rotate the arc into a fixed plane containing the axis,
(3) Straighten the resulting plane curve to the line segment.

2: Outside the cylinders, the curve has of $4\mu(C)$ segments. For each such segment, construct a tube neighborhood of radius small enough that no two tubes intersect. Inside each tube, isotope the curve segment to a straight line segment connecting its endpoints.

3: The output curve P is a polygon inscribed in C (up to the initial isotopy within cylinders). Recall that all isotopies can be chosen to be ambient and to fix the curve outside specified regions, P is equivalent to C by isotopy.

Also, note that $\mu(P, b) = \mu(C, b = \mu(C)$. ☐

Corollary 4.6. *A simple curve type G is tame if and only if $\mu(G) < \infty$.*

Proof. If G is tame, it will contain a polygon P, and $\mu(P) < \infty$. On the other side, if $C \in G$ with $\mu(C) < \infty$, then by Theorem 4.5, there is a polygon P equivalent to C, so G is tame. ☐

We now derive the main results we are dreaming for:

Theorem 4.7. *The crookedness of any knotted curve satisfies $\mu(C) \geq 2$. Equivalently, for any knot type G, we have $\mu(G) \geq 2$.*

Proof. Suppose, for contradiction, that there exists a knotted curve C with $\mu(C) = 1$. Let b be a direction achieving this minimum.

We use a similiar construction as Theorem 4.5 with $\mu(C, b) = 1$, we obtain only two cylinders. These cylinders can be arranged to share a common base, and the isotopy transforms C into a quadrilateral lying in a plane. But any plane quadrilateral is unknotted. This is a contradiction to the assumption that C is knotted.

So, $\mu(C) \geq 2$. ☐

Theorem 4.8 (Fary Milnor Theorem)**.** *The total curvature of any knotted curve satisfies $\kappa(C) \geq 4\pi$. Equivalently, for any knot type G, we have $\kappa(G) \geq 4\pi$.*

Proof. By Theorem 4.7, any knotted curve has $\mu(C) \geq 2$. From Theorem 3.7 (the curvature-crookedness inequality), we have:

$$\kappa(C) \geq 2\pi\mu(C) \geq 4\pi.$$

Taking the infimum over all curves in a knot type G gives $\kappa(G) \geq 4\pi$. ☐