MATH 4901 Final Report

Proofs to the Fary-Milnor Theorem

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1. Introduction

We know from Do Carmo's Differential Geometry of Curves and Surfaces textbook that in a $\mathbb{R}^2$ plane, a simple closed curve has curvature $2\pi I$, where I denotes the rotation index. This result is easily obtained under the Frenet Frames:

For a simple closed plane curve, paraemtrize it using arclength. $\alpha : [0, l] \to \mathbb{R}^2$.

Then by the Frenet-Serret Formulas and note T to be the tangent vector:

$$\alpha''(s) = \frac{dT}{ds} = kn$$

where k is the curvature and n is the normal vector.

We introduct an angle function $\theta(s) \in [0, 2\pi]$ to be the angle that T makes with the x-axis. We can perform a translation to relocated the tangent vector T to the origin, and by the nature of arclength parametrization, T has unit length and all T on α is contained in a unit circle.

By geometric definition, $\theta(s) = \arctan \frac{y'}{x'}$, where $x' = \cos\theta(s)$, $y' = \sin\theta(s)$.

Then we are able to express the unit tangent vector $T = (\cos\theta(s), \sin\theta(s))$, and $\frac{dT}{ds}$ is:

$$\frac{dT}{ds} = \theta'(-\sin\theta, \cos\theta) = \theta' n$$

This and the previous result combined togehter gives $\theta' = k$, and θ can be re-defined to be $\theta : [0, l] \to R$:

$$\theta(s) = \int_0^s k(s) ds$$

This result explains that we can measure the total rotation of the curve α by integrating the curvature along the curve.

This definition for total rotation using curvature can be brought to $\mathbb{R}^3$, and Fary-Milnor Theorem states that if a is any closed curve in Euclidan space, and curvature is defined for every point on a, then if $\int_a |k(s)| ds \leq 4\pi$, then a is an unknot.

But before we prove it, we need to define what is an unknot.

Definition 1.1. The unknot (or trivial knot) is a simple closed polygonal curve in $\mathbb{R}^3$.

In contrast, a nontrivial knot is any knot that cannot be deformed into the unknot.

We define the following two operations on the triangular isotopies of knots:

(1) Add a vertex: Assume [p,q] is some edge of the the knot, choose x such that the solid triangle $\blacktriangle pqx$ does not intersect with any existing parts of the knot. Then we can erase [p,q] and add [p,x], [x,q] to the knot.
(2) Remove a vertex: This is the reverse operation of (1). If the solid triangle $\blacktriangle pqx$ does not intersect with any existing parts of the knot, we can erase [p,x], [x,q] and add [p,q] to the knot.

Triangles are polygons, so we can refine our definition here to be:

Definition 1.2. An unknot is a knot that is isotopic to a triangle, and a nontrivial knot is a knot that is not isotopic to a triangle.

2. Proof By John Milnor

First we will introduce the proof by John Milnor because he is one of people that has name on the theorem.

We start with defining the height function $h : (x, y, z) \to z$.

Proposition 2.1. *If h has only one local maximum on a simple closed polygonal curve α, and all other vertices are have different height values, α is a trivial knot.*

Proof. Let $\alpha = p_1, p_2 \dots p_n$. If $n < 4$, α cannot be a nontrivial knot. So we will mainly consider cases for $n \geq 4$.

For α, pick three vertices with greatest height, one of which is the local maximum. In order to preserve the uniqueness of the local maximum, the other two points we chose has to be on both sides of the maximum.

Note the highest p_{k-1}, p_k, p_{k+1}, where p_k is the local maximum. We can take modulus to apply this to p_{k-1}, p_k, and 1 also. So the loop structure would not be a problem.

We focus on the solid triagnle $\blacktriangle p_{k-1}p_kp_{k+1}$. Note that besides p_k, p_{k-1} and p_{k+1} is also connected to other points, call them p_{k-2} and p_{k+2}. We assumed α to be simple at the beginning, so none of $[p_{k-2}, p_{k-1}]$ can intersect any edges of the triangle. So we can iteratively appeal to operation (2) defined in the introduction for n-3 times to reduce α to be a triagnle. So we conclude that α is a trivial knot. □

Proof. Milnor's Proof to the Fary-Milnor Theorem:

Let $\alpha = p_1 \dots p_n$. Assume α is closed, so $p_n = p_0$. For such polygonal curves, the curvature at each vertex is defined to be the exterior angle. So if we try to sum the exterior angle to find total curvature of inscribed polygon, it might be difficult. However, if we are able to connect the exterior angle with area, the computation will be much easier.

We can use the surface area of a spherical lune to connect angle with surface area. We know that the surface area of a spherical lune is $2\theta\mathbb{R}^2$, where θ is the angle and R is the arclength. For our case, since we are only interested in total rotation, we can normalize R to be 1. Then we have a nice connection $S = 2\theta$, if we are able to calculate the surface area for the spherical lune.

We do not have to do this. Instead we can consider the set U_i formed by all unit vectors u as a whole. Then $u \in U_i$ if and only if the function $x \to < u, x >$ has a local maximum at p_i on α where $<, >$ denotes the scalar product.

We construct the axis with z-axis in the same direction with u. Then by Proposition 1 we proved earlier, $p \to < u, p >$ has at least two local maxima on α. Then the spherical lunes of the sets $U_1, \ldots, U_n$ sweeps over the unit sphere at least twice.

This result makes a successful connection between the total angle and the surface area of the unit sphere. The surface area of the unit sphere is 4π. Recalling that $S = 2\theta$ and the fact that the lunes corresponding to the sets sweeps the sphere at least twice:

$$\Phi(\alpha) = \frac{1}{2}(\text{Area}U_1, \text{Area}U_2, \ldots, \text{Area}U_n) \geq 4\pi$$

$\square$