Proving Liouville's Theorem Using PDE methods

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Intro

I am taking Math 4220/5220 Applied Complex analysis with Prof.Strogatz this semester. The extra credit question in preliminary exam was to prove Liouville's Theorem. Prof.Strogatz wanted us to use Cauchy's Theorem to prove this, but I saw "harmonic" and wondered whether there is a proof using PDE tools. Fortunately, Liouville's Thorem is a direct corollary to Strong Maximum Principle, which is a direct corollary to Mean Value Inequality, which is a corollary to Divergence Theorem. We will address them one by one below.

Setup

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $u \in C^2(\Omega)$. The Laplacian of u is defined as

$$\Delta u = \sum_{i=1}^{n} D_{ii} u = \operatorname{div}(\nabla u).$$

We say that:

$$\Delta u = 0 \implies u \text{ is harmonic in } \Omega.$$

If $\Delta u \geq 0$, then u is subharmonic, and if $\Delta u \leq 0$, then u is superharmonic.

The Divergence Theorem

For a bounded domain Ω in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$, and for a vector field $w \in C^1(\overline{\Omega})$, the Divergence Theorem states:

$$\int_{\Omega} \operatorname{div}(w)\, dx = \int_{\partial\Omega} w \cdot \nu \, ds,$$

where ν is the outward unit normal to $\partial\Omega$ and ds is the $(n-1)$-dimensional surface measure.

In particular, taking $w = \nabla u$, we have:

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, ds.$$

Mean Value Inequalities

For any ball $B = B_R(y) \subset \Omega$, we have the mean value property for harmonic functions:

$$u(y) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B} u \, ds = \frac{1}{\omega_n R^n} \int_B u \, dx,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$ (i.e., $\omega_n = |B_1(0)|$).

Proof of the Mean Value Inequalities

Let $0 < \rho < R$ and consider $B_\rho(y)$. Applying the identity

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} \, ds = \int_{B_\rho} \Delta u \, dx,$$

and noting that for harmonic u, $\Delta u = 0$, we find:

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} \, ds = 0.$$

Introducing radial and angular coordinates, $r = |x - y|$, and writing $u(x) = u(y + r\omega)$ with $\omega \in S^{n-1}$, we have:

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} ds = \int_{|\omega|=1} \frac{\partial u}{\partial r}(y + \rho\omega)\rho^{n-1} d\omega = \rho^{n-1} \frac{d}{d\rho} \left(\int_{|\omega|=1} u(y + \rho\omega) d\omega \right) = 0.$$

This implies that the integral mean (over the sphere) of u is constant with respect to ρ. Taking the limit as $\rho \to 0$, we get the mean value property at the center y:

$$u(y) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(y)} u \, ds.$$

Integrating this relation with respect to ρ from 0 to R yields the solid mean value formula:

$$u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u \, dx.$$

These results show that a harmonic function attains its value at a point as the average of its values on any sphere or ball centered at that point.

Maximum and Minimum Principles

Theorem 1 (Maximum Principle). *Let $\Delta u \geq 0$ in Ω. Suppose there exists a point $y \in \Omega$ such that $u(y) = \sup_\Omega u$. Then u is constant in Ω. Similarly, if $\Delta u \leq 0$ and there is a point y such that $u(y) = \inf_\Omega u$, then u is constant. Consequently, a harmonic function (with $\Delta u = 0$) cannot have a non-constant interior maximum or minimum.*

Proof. Assume $\Delta u \geq 0$ in Ω. Let $M = \sup_\Omega u$ and define $\Omega_M = \{x \in \Omega \mid u(x) = M\}$. By assumption, Ω_M is not empty. Since u is continuous, Ω_M is closed relative to Ω. Take any point $z \in \Omega_M$. Consider $u - M$ in a ball $B = B_R(z) \subset \Omega$. By the mean value inequality for subharmonic functions,

$$0 = u(z) - M \leq \frac{1}{\omega_n R^n} \int_B (u - M)\, dx \leq 0.$$

This forces $u = M$ in $B_R(z)$, hence Ω_M is also open relative to Ω. Thus $\Omega_M = \Omega$, and so u is constant.

For superharmonic functions, the argument is similar, replacing u by $-u$. □

Liouville's Theorem

Liouville's theorem states that a harmonic function $u : \mathbb{R}^n \to \mathbb{R}$ that is bounded above (or below) must be a constant function. The proof is a consequence of applying the maximum (or minimum) principle to entire harmonic functions.

Proof. Assume u is harmonic.$\Delta u = 0$, and u is bounded above by M.

Create auxilary function $v(x) = M - u(x)$. By construction $v(x) \geq 0$, $\Delta v = 0$.

Assume, for the sake of proof by contradiction, that there exists a point in a ball sitting in Ω around arbitrary point x_0. And there exists a point x_1 such that $v(x_1) \neq v(x_0)$.

Then we can discuss which is larger, in either cases, a maximum has to appear, triggering maximum principle and making v constant equal to zero. So u = M for every points. □

References

Gilbarg, D., Trudinger, N. S. (1998). Elliptic partial differential equations of second order. Springer.