MATH 6390 PRESENTATION NOTES: LIE GROUPS AND RICCI FLOW

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1. Intro

Consider the Ricci flow equation:

$$\frac{\partial g}{\partial t} = -2\mathrm{Rc}(g). \tag{1.1}$$

For example, we write it out in 3 dimensional euclidean frame.

$$\begin{aligned}\mathrm{Rc}_{ij} =& \frac{1}{2} g^{kl}\left(\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}\right)\\ &+ g^{kl} g_{pq}\left(\Gamma^p_{il}\Gamma^q_{jk} - \Gamma^p_{kl}\Gamma^q_{ij}\right),\end{aligned}$$

where the Christoffel symbols are:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

So equation 1.1 is a system of 6 coupled and nonlinear partial differential equations. Which is impossible to be solved directly.

2. Lie Groups and Left-Invariant Metrics

Definition 2.1. A Lie group G is a C^∞ manifold that also has the structure of a group, such that the group operations:

$$\mu : G \times G \to G, \quad \mu(\sigma, \tau) = \sigma \cdot \tau^{-1}$$

are C^∞ maps.

For any $\sigma \in G$, we define:

- Left translation: $\sigma_L : G \to G$, $\sigma_L(\tau) = \sigma \cdot \tau$,
- Right translation: $\sigma_R : G \to G$, $\sigma_R(\tau) = \tau \cdot \sigma$.

These maps are diffeomorphisms of G.

Definition 2.2. A vector field X on G is called left-invariant if for every $\sigma \in G$:

$$(\sigma_L)_* X = X.$$

This is just the pushforward of X by left translation equals X itself.

A left-invariant vector field is determined by its value at the identity $e \in G$. In other words, if $X_e \in T_e G$, then we can define:

$$X_\sigma = (\sigma_L)_* X_e.$$

This gives a one-to-one correspondence between left-invariant vector fields and tangent vectors at the identity.

Definition 2.3. A Riemannian metric g on G is called **left-invariant** if for every $\sigma \in G$, left translation σ_L is an isometry:

$$(\sigma_L)^* g = g.$$

A left-invariant metric is determined by an inner product on the Lie algebra $L = T_e G$. For example, if we fix a basis $\{E_1, \ldots, E_n\}$ for L, then the metric at any point is determined by the constant matrix:

$$g_{ij} = \langle E_i, E_j \rangle.$$

Recall in Riemannian geometry, we have a unique torsion-free, metric-compatible connection - the Levi-Civita connection. For general vector fields, this is given by the Koszul formula:

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\quad - \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Y,Z], X\rangle. \end{aligned} \tag{2.1}$$

Lemma 2.4. *If X, Y, Z are left-invariant vector fields on a Lie group with a left-invariant metric, then the function $\langle Y, Z\rangle$ is constant. Consequently:*

$$X\langle Y, Z\rangle = Y\langle Z, X\rangle = Z\langle X, Y\rangle = 0.$$

Proof. Since Y and Z are left-invariant and the metric is left-invariant, for any $\sigma \in G$:

$$\langle Y_\sigma, Z_\sigma\rangle = \langle (\sigma_L)_* Y_e, (\sigma_L)_* Z_e\rangle = \langle Y_e, Z_e\rangle,$$

where the last equality follows because σ_L is an isometry. Thus $\langle Y, Z\rangle$ is constant. □

Applying this to the Koszul formula 2.1:

Lemma 2.5. *If X, Y, Z are left-invariant vector fields on a Lie group with a left-invariant metric, then:*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle\right). \tag{2.2}$$

Proof.

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= -\langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Y, Z], X\rangle \\ &= \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle, \end{aligned}$$

where in the last step we used the antisymmetry of the Lie bracket and the symmetry of the inner product. □

The Levi-Civita connection now can be computed using only the Lie bracket and the inner product on the Lie algebra L.

Definition 2.6. The Riemanian curvature tensor is defined by:

$$\mathrm{Rm}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Substitute our connection formula directly into this definition:

Lemma 2.7. *For left-invariant vector fields X, Y, Z, W:*

$$\langle \mathrm{Rm}(X, Y)Z, W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle. \tag{2.3}$$

Proof. By definition:

$$\langle \mathrm{Rm}(X, Y)Z, W\rangle = \langle \nabla_X \nabla_Y Z, W\rangle - \langle \nabla_Y \nabla_X Z, W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle.$$

Use the metric compatibility of the connection:

$$\langle \nabla_X \nabla_Y Z, W\rangle = X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle,$$

and similarly for the other term. Since $\langle \nabla_Y Z, W\rangle$ is constant (as all vector fields are left-invariant), its derivative vanishes. □

For sectional curvature computations, the most useful form is:

Lemma 2.8. *For left-invariant vector fields X, Y:*

$$\langle \mathrm{Rm}(X,Y)Y, X\rangle = \langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle - \langle \nabla_{[X,Y]} Y, X\rangle. \tag{2.4}$$

Definition 2.9. A Riemannian metric on a Lie group is called bi-invariant if it is both left-invariant and right-invariant.

Bi-invariant metrics have particularly elegant properties. First, we need a technical lemma:

Lemma 2.10. *If g is a bi-invariant metric on G, then every left-invariant vector field is a killing vector field.*

Proof. A vector field X is Killing if the Lie derivative of the metric with respect to X vanishes: $\mathscr{L}_X g = 0$. For a left-invariant vector field on a group with bi-invariant metric, this follows from the fact that the flow of X consists of right translations, which are isometries. ☐

Lemma 2.11 (Connection Formula for Killing Fields)**.** *If X, Y, Z are Killing vector fields, then:*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle [X,Y], Z\rangle + \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle\right). \tag{2.5}$$

Proof. Since Y is Killing, we have:

$$\langle \nabla_X Y, Z\rangle + \langle \nabla_Z Y, X\rangle = 0.$$

Similarly, since X and Z are Killing:

$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0, \quad \langle \nabla_X Z, Y\rangle + \langle \nabla_Y Z, X\rangle = 0.$$

Now consider the sum:

$$\begin{aligned}
&\langle [X,Y], Z\rangle + \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle \\
&= \langle \nabla_X Y - \nabla_Y X, Z\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle + \langle \nabla_Y Z - \nabla_Z Y, X\rangle \\
&= \langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle + \langle \nabla_X Z, Y\rangle - \langle \nabla_Z X, Y\rangle + \langle \nabla_Y Z, X\rangle - \langle \nabla_Z Y, X\rangle.
\end{aligned}$$

Rearranging and using the Killing field identities:

$$\begin{aligned}
&= \langle \nabla_X Y, Z\rangle + \langle \nabla_X Z, Y\rangle + \langle \nabla_Y Z, X\rangle - \langle \nabla_Y X, Z\rangle - \langle \nabla_Z X, Y\rangle - \langle \nabla_Z Y, X\rangle \\
&= \langle \nabla_X Y, Z\rangle - \langle \nabla_X Z, Y\rangle + \langle \nabla_Y Z, X\rangle + \langle \nabla_Y X, Z\rangle - \langle \nabla_Z X, Y\rangle + \langle \nabla_Z Y, X\rangle \\
&= 2\langle \nabla_X Y, Z\rangle.
\end{aligned}$$

☐

Lemma 2.12. *Let g be a bi-invariant metric on G. If X, Y, Z, W are left-invariant vector fields, then:*

(1) The Levi-Civita connection is given by:

$$\nabla_X Y = \frac{1}{2}[X, Y]. \tag{2.6}$$

(2) The Riemann curvature tensor satisfies:

$$\langle \mathrm{Rm}(X, Y)Z, W\rangle = \frac{1}{4}\left(\langle [X, W], [Y, Z]\rangle - \langle [X, Z], [Y, W]\rangle\right). \tag{2.7}$$

Proof. (1) For a bi-invariant metric, we have both formulas (2) and (5). Adding them together:

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle + 2\langle \nabla_X Y, Z\rangle &= \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle \\ &\quad + \langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle \\ &= 2\langle [X, Y], Z\rangle. \end{aligned}$$

Thus $\langle \nabla_X Y, Z\rangle = \frac{1}{2}\langle [X, Y], Z\rangle$ for all Z, which implies $\nabla_X Y = \frac{1}{2}[X, Y]$.

(2) Using the connection formula and the definition of curvature:

$$\begin{aligned} \mathrm{Rm}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z]. \end{aligned}$$

Now use the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

which implies:

$$[X, [Y, Z]] - [Y, [X, Z]] = [[X, Y], Z].$$

Substituting this in:

$$\mathrm{Rm}(X, Y)Z = \frac{1}{4}[[X, Y], Z] - \frac{1}{2}[[X, Y], Z] = -\frac{1}{4}[[X, Y], Z].$$

Taking the inner product with W:

$$\langle \mathrm{Rm}(X, Y)Z, W\rangle = -\frac{1}{4}\langle [[X, Y], Z], W\rangle.$$

Now use the following property of the metric:

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle,$$

to get:

$$\begin{aligned}\langle[[X,Y],Z],W\rangle &= -\langle Z,[[X,Y],W]\rangle\\ &= -\langle[X,Y],[Z,W]\rangle\\ &= -\langle[X,W],[Y,Z]\rangle + \langle[X,Z],[Y,W]\rangle,\end{aligned}$$

□

Corollary 2.13. *A bi-invariant metric on a Lie group has nonnegative sectional curvature.*

Proof. For sectional curvature, take $Z = Y$ and $W = X$:

$$\begin{aligned}\langle \mathrm{Rm}(X,Y)Y,X\rangle &= \frac{1}{4}\left(\langle[X,X],[Y,Y]\rangle - \langle[X,Y],[Y,X]\rangle\right)\\ &= -\frac{1}{4}\langle[X,Y],[Y,X]\rangle\\ &= \frac{1}{4}\|[X,Y]\|^2 \geq 0.\end{aligned}$$

□

3. Applications: Ricci Flow on 3D Lie Groups

Now we apply the theoretical machinery to study the Ricci flow on 3-dimensional unimodular Lie groups. The symmetry reduces the PDE to a system of ODEs.

Definition 3.1. A Lie group G is called unimodular if its volume form is bi-invariant. For 3-dimensional unimodular Lie groups, there exists a special left-invariant frame called the Milnor frame.

Let $\{f_1, f_2, f_3\}$ be a Milnor frame with dual coframe $\{\eta^1, \eta^2, \eta^3\}$ such that the Lie brackets satisfy:

$$[f_2, f_3] = \lambda f_1, \quad [f_3, f_1] = \mu f_2, \quad [f_1, f_2] = \nu f_3, \tag{3.1}$$

where $\lambda, \mu, \nu \in \{0, \pm 2\}$.

Consider a left-invariant metric that is diagonal in this frame:

$$g = A\eta^1 \otimes \eta^1 + B\eta^2 \otimes \eta^2 + C\eta^3 \otimes \eta^3, \tag{3.2}$$

with $A, B, C > 0$. The orthonormal frame is given by:

$$e_1 = \frac{1}{\sqrt{A}} f_1, \quad e_2 = \frac{1}{\sqrt{B}} f_2, \quad e_3 = \frac{1}{\sqrt{C}} f_3.$$

Lemma 3.2. *In the orthonormal frame $\{e_1, e_2, e_3\}$, the Lie brackets become:*

$$[e_2, e_3] = \frac{\lambda\sqrt{A}}{\sqrt{ABC}} e_1,$$
$$[e_3, e_1] = \frac{\mu\sqrt{B}}{\sqrt{ABC}} e_2,$$
$$[e_1, e_2] = \frac{\nu\sqrt{C}}{\sqrt{ABC}} e_3.$$

Theorem 3.3. *The sectional curvatures are given by:*

$$K(e_2 \wedge e_3) = \frac{(\mu B - \nu C)^2}{4ABC} + \lambda \frac{2\mu B + 2\nu C - 3\lambda A}{4BC},$$
$$K(e_3 \wedge e_1) = \frac{(\nu C - \lambda A)^2}{4ABC} + \mu \frac{2\nu C + 2\lambda A - 3\mu B}{4AC},$$
$$K(e_1 \wedge e_2) = \frac{(\lambda A - \mu B)^2}{4ABC} + \nu \frac{2\lambda A + 2\mu B - 3\nu C}{4AB}.$$

Proof. We compute using Equation 2.7.

For example, for $K(e_2 \wedge e_3) = \langle \mathrm{Rm}(e_2, e_3)e_3, e_2 \rangle$, we need to compute:

$$\langle \nabla_{e_2} e_3, \nabla_{e_3} e_2 \rangle - \langle \nabla_{e_3} e_3, \nabla_{e_2} e_2 \rangle - \langle \nabla_{[e_2,e_3]} e_3, e_2 \rangle.$$

Using the connection formula and the Lie brackets from the previous lemma, we obtain the result. See Appendix A. □

Corollary 3.4. *The Ricci tensor is diagonal in the* $\{e_i\}$ *frame:*

$$Rc(e_1, e_1) = \frac{(\lambda A)^2 - (\mu B - \nu C)^2}{2ABC}, \tag{3.3}$$

$$Rc(e_2, e_2) = \frac{(\mu B)^2 - (\nu C - \lambda A)^2}{2ABC}, \tag{3.4}$$

$$Rc(e_3, e_3) = \frac{(\nu C)^2 - (\lambda A - \mu B)^2}{2ABC}. \tag{3.5}$$

Proof. Since the frame is orthonormal, $\mathrm{Rc}(e_i, e_j) = \sum_k \langle \mathrm{Rm}(e_i, e_k)e_k, e_j\rangle$. The formulas follow from summing the appropriate sectional curvatures. □

The unnormalized Ricci flow $\frac{\partial g}{\partial t} = -2\mathrm{Rc}(g)$ gives us:

Theorem 3.5. *The metric coefficients evolve according to:*

$$\frac{dA}{dt} = \frac{(\mu B - \nu C)^2 - (\lambda A)^2}{BC}, \tag{3.6}$$

$$\frac{dB}{dt} = \frac{(\nu C - \lambda A)^2 - (\mu B)^2}{AC}, \tag{3.7}$$

$$\frac{dC}{dt} = \frac{(\lambda A - \mu B)^2 - (\nu C)^2}{AB}. \tag{3.8}$$

Proof. Since $g(e_1, e_1) = 1$, we have $\frac{\partial}{\partial t} g(e_1, e_1) = 0$. However, the metric coefficients A, B, C in the Milnor frame satisfy:

$$\frac{dA}{dt} = -2\mathrm{Rc}(e_1, e_1), \quad \frac{dB}{dt} = -2\mathrm{Rc}(e_2, e_2), \quad \frac{dC}{dt} = -2\mathrm{Rc}(e_3, e_3).$$

Substituting equations (3.3)-(3.5) gives the result. □

Example 3.6 (Ricci Flow on SU(2)). For spherical unitary group $SU(2)$, we have $\lambda = \mu = \nu = -2$. The Ricci flow equations become:

$$\begin{aligned}
\frac{dA}{dt} &= \frac{4(B-C)^2 - 4A^2}{BC} = 4\frac{(B-C)^2 - A^2}{BC}, \\
\frac{dB}{dt} &= 4\frac{(C-A)^2 - B^2}{AC}, \\
\frac{dC}{dt} &= 4\frac{(A-B)^2 - C^2}{AB}.
\end{aligned}$$

For the normalized Ricci flow (where we maintain constant volume), the equations become:

$$\frac{dA}{dt} = A\left(A(B+C-2A) + (B-C)^2\right),$$
$$\frac{dB}{dt} = B\left(B(A+C-2B) + (A-C)^2\right),$$
$$\frac{dC}{dt} = C\left(C(A+B-2C) + (A-B)^2\right).$$

Theorem 3.7. *For any left-invariant initial metric g_0 on $SU(2)$, the normalized Ricci flow converges to a constant positive sectional curvature metric as $t \to \infty$.*

Sketch of Proof. Assume without loss of generality that $A(0) \geq B(0) \geq C(0)$. We can show:

(1) $C(t)$ is non-decreasing and bounded below by $C(0) > 0$
(2) $A(t) - C(t)$ decays exponentially: $A(t) - C(t) \leq (A(0) - C(0))e^{-kt}$ for some $k > 0$
(3) By volume normalization (ABC constant), all three functions converge to the same positive limit

Thus the metric converges to a round metric. □

Example 3.8 (Ricci Flow on the Heisenberg Group (Nil)). For the Heisenberg group, we have $\lambda = -2$, $\mu = \nu = 0$. The Ricci flow equations simplify dramatically:

$$\frac{dA}{dt} = \frac{-4A^2}{BC} = -4\frac{A^2}{BC},$$
$$\frac{dB}{dt} = \frac{4A^2}{AC} = 4\frac{A}{C},$$
$$\frac{dC}{dt} = \frac{4A^2}{AB} = 4\frac{A}{B}.$$

Theorem 3.9. *The solution exists for all time and exhibits collapse: the diameters grow like $t^{1/6}$ while the sectional curvatures decay like t^{-1}, so $|sec| \cdot diam^2 \to 0$ as $t \to \infty$.*

Proof. We can solve the equations explicitly. Notice that:

$$\frac{d}{dt}\log A = -4\frac{A}{BC}, \quad \frac{d}{dt}\log B = 4\frac{A}{BC}, \quad \frac{d}{dt}\log C = 4\frac{A}{BC}.$$

Thus B/C, AB, and AC are constant. We find:

$$\frac{A}{BC}(t) = \frac{1}{12}\left(\frac{B_0C_0}{12A_0} + t\right)^{-1},$$

and the metric coefficients satisfy:

$$\frac{A_0}{A(t)} = \frac{B(t)}{B_0} = \frac{C(t)}{C_0} = \left(1 + \frac{12A_0}{B_0C_0}t\right)^{1/3}.$$

□

Thus the solution collapses as time tends to infinity(The metric become more and more alomst flat).

Appendix A. Calculations of sectional curvature

Calculate $K(e_2 \wedge e_3) = \langle \mathrm{Rm}(e_2, e_3)e_3, e_2\rangle$ using the formula:

$$\langle \mathrm{Rm}(X,Y)Y,X\rangle = \langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle - \langle \nabla_{[X,Y]}Y, X\rangle. \tag{A.1}$$

Set $X = e_2$ and $Y = e_3$. We'll compute each term separately.

A.0.1. $\nabla_{e_2}e_3$ *and* $\nabla_{e_3}e_2$. Using the connection formula for left-invariant vector fields:

$$\langle \nabla_{e_i}e_j, e_k\rangle = \frac{1}{2}\left(\langle [e_i,e_j],e_k\rangle - \langle [e_i,e_k],e_j\rangle - \langle [e_j,e_k],e_i\rangle\right),$$

we compute:

$$\begin{aligned}\langle \nabla_{e_2}e_3, e_1\rangle &= \frac{1}{2}\left(\langle [e_2,e_3],e_1\rangle - \langle [e_2,e_1],e_3\rangle - \langle [e_3,e_1],e_2\rangle\right)\\ &= \frac{1}{2}\left(\frac{\lambda A}{D} - \left(-\frac{\nu C}{D}\right) - \frac{\mu B}{D}\right)\\ &= \frac{1}{2D}(\lambda A + \nu C - \mu B),\end{aligned}$$

where $D = \sqrt{ABC}$. The other components vanish:

$$\begin{aligned}\langle \nabla_{e_2}e_3, e_2\rangle &= 0,\\ \langle \nabla_{e_2}e_3, e_3\rangle &= 0.\end{aligned}$$

So:

$$\nabla_{e_2}e_3 = \frac{1}{2D}(\lambda A + \nu C - \mu B)e_1.$$

Similarly:

$$\begin{aligned}\langle \nabla_{e_3}e_2, e_1\rangle &= \frac{1}{2}\left(\langle [e_3,e_2],e_1\rangle - \langle [e_3,e_1],e_2\rangle - \langle [e_2,e_1],e_3\rangle\right)\\ &= \frac{1}{2}\left(-\frac{\lambda A}{D} - \frac{\mu B}{D} + \frac{\nu C}{D}\right)\\ &= \frac{1}{2D}(-\lambda A - \mu B + \nu C),\end{aligned}$$

and other components vanish. Thus:

$$\nabla_{e_3}e_2 = \frac{1}{2D}(-\lambda A - \mu B + \nu C)e_1.$$

A.0.2. $\langle \nabla_{e_2} e_3, \nabla_{e_3} e_2 \rangle$.

$$\begin{aligned}\langle \nabla_{e_2} e_3, \nabla_{e_3} e_2 \rangle &= \left(\frac{1}{2D}\right)^2 (\lambda A + \nu C - \mu B)(-\lambda A - \mu B + \nu C) \\ &= -\frac{1}{4ABC}(\lambda A + \nu C - \mu B)(\lambda A + \mu B - \nu C).\end{aligned}$$

A.0.3. $\langle \nabla_{e_3} e_3, \nabla_{e_2} e_2 \rangle$. We claim that $\nabla_{e_i} e_i = 0$ for any i. For $i = 2$:

$$\begin{aligned}&\langle \nabla_{e_2} e_2, e_1 \rangle = -\langle [e_2, e_1], e_2 \rangle = 0, \\ &\langle \nabla_{e_2} e_2, e_2 \rangle = 0, \\ &\langle \nabla_{e_2} e_2, e_3 \rangle = -\langle [e_2, e_3], e_2 \rangle = 0.\end{aligned}$$

Similarly, $\nabla_{e_3} e_3 = 0$. So:

$$\langle \nabla_{e_3} e_3, \nabla_{e_2} e_2 \rangle = 0.$$

A.0.4. $\langle \nabla_{[e_2,e_3]} e_3, e_2 \rangle$. First, $[e_2, e_3] = \frac{\lambda A}{D} e_1$.

$$\begin{aligned}\langle \nabla_{e_1} e_3, e_2 \rangle &= \frac{1}{2}\left(\langle [e_1, e_3], e_2 \rangle - \langle [e_1, e_2], e_3 \rangle - \langle [e_3, e_2], e_1 \rangle\right) \\ &= \frac{1}{2}\left(-\frac{\mu B}{D} - \frac{\nu C}{D} + \frac{\lambda A}{D}\right) \\ &= \frac{1}{2D}(\lambda A - \mu B - \nu C).\end{aligned}$$

Thus:

$$\nabla_{e_1} e_3 = \frac{1}{2D}(\lambda A - \mu B - \nu C) e_2.$$

Then:

$$\begin{aligned}\nabla_{[e_2,e_3]} e_3 &= \frac{\lambda A}{D} \nabla_{e_1} e_3 \\ &= \frac{\lambda A}{D} \cdot \frac{1}{2D}(\lambda A - \mu B - \nu C) e_2 \\ &= \frac{\lambda A}{2ABC}(\lambda A - \mu B - \nu C) e_2.\end{aligned}$$

So

$$\langle \nabla_{[e_2,e_3]} e_3, e_2 \rangle = \frac{\lambda A}{2ABC}(\lambda A - \mu B - \nu C).$$

A.0.5. *Combine All Terms.*

$$\begin{aligned} K(e_2 \wedge e_3) &= \langle \nabla_{e_2} e_3, \nabla_{e_3} e_2 \rangle - \langle \nabla_{e_3} e_3, \nabla_{e_2} e_2 \rangle - \langle \nabla_{[e_2,e_3]} e_3, e_2 \rangle \\ &= -\frac{1}{4ABC}(\lambda A + \nu C - \mu B)(\lambda A + \mu B - \nu C) \\ &\quad - \frac{\lambda A}{2ABC}(\lambda A - \mu B - \nu C). \end{aligned}$$

Let $X = \lambda A$, $Y = \mu B$, $Z = \nu C$. Then:

$$\begin{aligned} K(e_2 \wedge e_3) &= -\frac{1}{4ABC}\left[(X + Z - Y)(X + Y - Z) + 2X(X - Y - Z)\right] \\ &= -\frac{1}{4ABC}\left[X^2 - Y^2 - Z^2 + 2YZ + 2X^2 - 2XY - 2XZ\right] \\ &= -\frac{1}{4ABC}\left[3X^2 - Y^2 - Z^2 - 2XY - 2XZ + 2YZ\right] \\ &= \frac{-3\lambda^2 A^2 + \mu^2 B^2 + \nu^2 C^2 + 2\lambda\mu AB + 2\lambda\nu AC - 2\mu\nu BC}{4ABC}. \end{aligned}$$

Rearranging we have:

$$K(e_2 \wedge e_3) = \frac{(\mu B - \nu C)^2}{4ABC} + \lambda\frac{2\mu B + 2\nu C - 3\lambda A}{4BC}.$$